ON SUMMABILITY OF DOUBLE SERIES*

BY

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1. **Introduction.** Although the theory of summability of simple series has been brought to a rather high degree of development, it is fair to say that the extension of this theory to multiple series is still in its infancy. In the present paper we view the question of summability from the standpoint of transformation of sequences and establish for double sequences a considerable number of analogues of well known theorems in the elementary theory of summability. We hope to return later to the problem of generalizing some of the more advanced developments of the theory, such as, for example, those due to Hausdorff.†

The Pringsheim definition of convergence of double series will be used, since it alone among the definitions commonly employed permits a series to converge conditionally. Under this definition a series $\sum_{i,j=1}^{\infty} u_{ij}$ is convergent if and only if the sequence of partial sums,

$$s_{mn} = \sum_{i=1,\,j=1}^{m,n} u_{ij},$$

converges; i.e., if the limit of s_{mn} exists as m and n become infinite *simultaneously but independently*. This manner of indefinite increase is to be understood whenever the symbol

$$\lim_{m,n\to\infty}$$

appears hereafter.

Let $\{x_{mn}\}$ be a double sequence and

$$\|a_{mnkl}\| \qquad (m,\ n,\ k,\ l = 1,\ 2,\ 3,\ \cdots)$$

a four-dimensional matrix of numbers, real or complex, with

$$a_{mnkl} = 0 \qquad \text{for } k > m \text{ or } l > n \text{ or both.}$$

Then the transformation

* The chief results of this paper, in essentially the form given here, were presented to the Society, September 8, 1931, under the title *Transformations of double sequences, with application to Cesàro summability of double series*. The paper was received by the editors November 9, 1931.

† Hausdorff, *Summationsmethoden und Momentfolgen*, I, II, Mathematische Zeitschrift, vol.9 (1921), pp. 74–109, 280–299.

$$y_{mn} = \sum_{k=1,l=1}^{m,n} a_{mnkl} x_{kl},$$

which we denote by A, defines a new double sequence $\{y_{mn}\}$. Clearly A is the analogue of the transformation of simple sequences defined by a triangular matrix.

In order that it may be useful as a method of summability it is natural to require a transformation A to be *regular for some class of sequences*, in the sense that it carry every convergent sequence $\{x_{mn}\}$ of that class into a sequence $\{y_{mn}\}$ convergent to the same limit. Necessary and sufficient conditions that A be regular for the class of *all* double sequences have been found by Kojima.* One might well expect that the class of transformations thus regular would be extremely restricted, since a double series can behave so very badly and yet converge. Such turns out to be precisely the case ,even the arithmetic mean transformation M, defined by

$$a_{mnkl} = 1/(mn),$$

being excluded from this class of transformations. It is desirable, therefore, to enlarge the class of transformations admitted to consideration, even at the expense of limiting the class of sequences for which the transformations are regular. Thus at the outset the theory of transformations of double sequences is markedly dissimilar from that of simple sequence transformations.

It has long since been observed that to a considerable extent *convergence plus boundedness* plays for double sequences a rôle analogous to that of convergence for simple sequences. There is little doubt, therefore, that the class of bounded convergent sequences is the most important sub-class of all convergent double sequences. Hence it is natural to concern oneself especially with the class of transformations A which are regular for the class of bounded sequences. Necessary and sufficient conditions that A be so regular have been found by Robison.† We have recently shown‡ that a transformation A, regular for bounded sequences, is in general regular for a much larger class of sequences.

A transformation A will be said to be the "product" of two transformations of simple sequences, A' and A'', defined respectively by matrices $\|a'_{mk}\|$ and $\|a''_{nl}\|$, when we have

* Kojima, *On the theory of double sequences*, Tôhoku Mathematical Journal, vol. 21 (1922), pp. 3–14.

† Robison, *Divergent double sequences and series*, Dissertation (Cornell), 1919; these Transactions, vol. 28 (1926), pp. 50–73.

‡ Adams, *Transformations of double sequences, with application to Cesàro summability of double series*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 741–748. This paper, which will be referred to hereafter as I, contains references to the literature of the subject.

$$a_{mnkl} = a'_{mk} \cdot a''_{nl} \qquad (m, n, k, l = 1, 2, 3, \cdots).$$

We then write

$$A = A' \odot A''$$

to distinguish this type of product from the ordinary product $A' \cdot A''$, which indicates the result of performing A'' upon a simple sequence and then A' upon its transform. For the ordinary product of two double sequence transformations we shall use the notation $B \cdot A$. In general throughout this paper unprimed capital letters will stand for transformations of double sequences, while primed capitals will be used for simple sequence transformations.

Double sequence transformations of the product type are of special interest and importance because they include the arithmetic mean transformation M and, so far as we are aware, all generalizations yet made for double sequences of the simple sequence transformations which are defined by triangular matrices and bear the names of Cesàro, Hölder, etc. Here we consider mainly transformations of the product type. Concerning them we have established in I the following theorem.

THEOREM 1A. *Let A' and A'' be any two regular transformations of simple sequences; then the transformation $A = A' \odot A''$ is regular for the class of double sequences of which each row is transformable by A'', and each column by A', into a bounded sequence.*

Almost simultaneously with I there appeared a paper by Lösch* in which a somewhat better result was independently obtained. His theorem may be stated in the following form.†

THEOREM 1L. *Let A' and A'' be any two regular transformations of simple sequences; then the transformation $A = A' \odot A''$ is regular for the class of double sequences which are bounded A.*‡

Many of the subsequent theorems in the present paper are based upon Theorem 1L; we can thus give them a more general and simpler form than would have been possible had they been made to depend upon Theorem 1A, as was our original intention.

Each of the Theorems 1A and 1L is valid when the factor transforma-

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* Lösch, *Über den Permanenzsatz gewisser Limitierungsverfahren für Doppelfolgen*, Mathematische Zeitschrift, vol. 34 (1931), pp. 281–290.

† It may be remarked that if the factor transformations A' and A'' are defined by matrices containing no zeros in their main diagonals, Theorems 1A and 1L give identical results.

‡ A sequence $\{x_{mn}\}$ will be said to be *bounded* A if its transform, $A\{x_{mn}\}$, is bounded.

tions, A' and A'', are defined by square rather than triangular matrices, provided certain further restrictions are made upon the class of sequences $\{x_{mn}\}$ involved. In particular it is first necessary to restrict this class to sequences for which every double series

$$\sum_{k,l=1}^{\infty} a'_{mk}\cdot a''_{nl}x_{kl} \qquad (m, n = 1, 2, 3, \cdots)$$

converges, in order that the sequence $\{y_{mn}\}$ may be completely defined; secondly, in order to apply the methods of proof already used in establishing the two theorems, it is necessary to assume that each of these double series has convergent rows and columns. Of course these conditions are satisfied by any bounded sequence, and they do not constitute real restrictions when A' and A'' are defined by row-finite, but not triangular, matrices. In the following pages it is to be understood that the simple sequence transformations involved are defined by triangular matrices; the results are always valid, however, for the case of row-finite matrices, and in general can be extended to the case of matrices which are not row-finite, unless the contrary is specified.

In §2 we consider a more general form of Theorem 1L and the question of whether the sufficient conditions for regularity therein contained are also necessary. In §3 is established an analogous theorem for convergence-preserving transformations. §4 is devoted to transformations defined by matrices of finite "rank" greater than unity. In §5 we consider the questions of inclusiveness and equivalence of two transformations. In §6 the adjunction or omission of a row or column is discussed. In §7 we establish certain sufficient conditions for mutual consistency of two transformations. §8 is devoted to the transformations defined by a particular kind of matrix of infinite "rank."

2. **An extension of Theorem 1L; necessity of the conditions.** First we state two lemmas concerning a pair of simple sequences, $\{a_m\}$ and $\{b_n\}$, of real or complex numbers. The proofs can readily be supplied by the reader.

LEMMA 1. *In order that we have*

$$\lim_{m,n\to\infty} a_m b_n = L \neq 0,$$

it is necessary and sufficient that $\lim_{m\to\infty} a_m$ *and* $\lim_{n\to\infty} b_n$ *both exist and their product equal L.*

LEMMA 2. *In order that we have*

$$\lim_{m,n\to\infty} a_m b_n = 0,$$

i is necessary that one of the sequences $\{a_m\}$, $\{b_n\}$ *converge to zero.*

Now an examination of the proof of Theorem 1L as given by Lösch discloses the fact that only the following hypotheses are actually used:

(1) $\lim_{m\to\infty} a'_{mk} = 0 \quad (k = 1, 2, 3, \cdots); \qquad \lim_{n\to\infty} a''_{nl} = 0 \qquad (l = 1, 2, 3, \cdots);$

(2) $\sum_{k=1}^{m} | a'_{mk} | < K, \quad \sum_{l=1}^{n} | a''_{nl} | < K \qquad (K = \text{constant}; m, n = 1, 2, 3, \cdots);$

and

$$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a'_{mk}a''_{nl} = 1.$$

By Lemma 1, this last condition is equivalent to the set of conditions

(3) $$\lim_{m\to\infty} \sum_{k=1}^{m} a'_{mk} = L_1, \quad \lim_{n\to\infty} \sum_{l=1}^{n} a''_{nl} = L_2,$$

(4) $$L_1 \cdot L_2 = 1.$$

It is well known that the combination of conditions (1), (2), and (3) is necessary and sufficient that the transformations A' and A'' be convergence-preserving and regular for null sequences.* Thus we obtain the following theorem which includes Theorem 1L.

THEOREM 2. *Let A' and A'' be any two transformations of simple sequences, each convergence-preserving and regular for null sequences, and let them satisfy the condition* (4); *then the transformation $A = A' \odot A''$ is regular for the class of double sequences which are bounded A.*

That the sufficient conditions for regularity given here are also, in a certain sense, necessary, we shall now see.

THEOREM 3. *Let $A = A' \odot A''$ be any transformation of the product type, regular for a class of double sequences which includes all bounded sequences; then each factor transformation is convergence-preserving and regular for null sequences, and together they satisfy condition* (4).

Since the transformation A is regular for all bounded sequences the following conditions must be fulfilled:†

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* Each of the matrices $\| a'_{mk} \|$ and $\| a''_{nl} \|$ is then a "pure C-matrix" in the language of Hausdorff, loc. cit., p. 75.

† See Robison, loc. cit., p. 53.

$$(5)\qquad \lim_{m,n\to\infty} a_{mnkl} = 0 \qquad (k, l = 1, 2, 3, \cdots),$$

$$(6)\qquad \lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{mnkl} = 1,$$

$$(7)\qquad \lim_{m,n\to\infty} \sum_{k=1}^{m} | a_{mnkl} | = 0 \qquad (l = 1, 2, 3, \cdots),$$

$$(8)\qquad \lim_{m,n\to\infty} \sum_{l=1}^{n} | a_{mnkl} | = 0 \qquad (k = 1, 2, 3, \cdots),$$

$$(9)\qquad \sum_{k=1,l=1}^{m,n} | a_{mnkl} | < K \qquad (K = \text{constant}; m, n = 1, 2, 3, \cdots).$$

From (6) it follows by Lemma 1 that the factor transformations satisfy the conditions (3) and (4). By (9) we have

$$\sum_{k=1}^{m} | a'_{mk} | \cdot \sum_{l=1}^{n} | a''_{nl} | < K \qquad (m, n = 1, 2, 3, \cdots).$$

Neither sum can vanish for all values of m or n without violating (3), (4); hence each sum is bounded, and conditions (2) are satisfied. From (7) we have

$$\lim_{m,n\to\infty} | a''_{nl} | \cdot \sum_{k=1}^{m} | a'_{mk} | = 0 \qquad (l = 1, 2, 3, \cdots).$$

The second factor does not tend to zero with $1/m$; therefore, by Lemma 2, A'' fulfills the second of conditions (1); that A' fulfills the first of (1) follows in a similar manner from (8).

3. **Convergence-preserving transformations of double sequences.** Necessary and sufficient conditions that the transformations A' and A'' be convergence-preserving are expressed by (2), (3), and*

$$(1')\quad \lim_{m\to\infty} a'_{mk} = \alpha'_k \quad (k = 1, 2, 3, \cdots); \quad \lim_{n\to\infty} a''_{nl} = \alpha''_l \quad (l = 1, 2, 3, \cdots).$$

Let the transformation defined by the matrix whose general element is

$$\bar{a}_{mnkl} = (a'_{mk} - \alpha'_k)(a''_{nl} - \alpha''_l)$$

be denoted by $\bar{A}$. From Theorem 1L is now easily obtained

THEOREM 4. *Let A' and A'' be any two convergence-preserving transformations of simple sequences. Then the transformation $A = A' \odot A''$ is convergence-*

* See, for example, Hausdorff, loc. cit., p. 75.

preserving for the class of double sequences $\{x_{mn}\}$ *which are bounded* $\bar{A}$ *and are such that the series*

$$\sum_{k,l=1}^{\infty}(\alpha_k' a_{nl}'' + \alpha_l'' a_{mk}' - \alpha_k' \alpha_l'')(x_{kl} - x), \tag{10}$$

where x is the limit of $\{x_{mn}\}$, *converges. Moreover, the A-transform of a convergent sequence satisfying these conditions converges to the limit*

$$L_1 L_2 x + S,$$

where S denotes the sum of the series (10).

4. **Transformations defined by matrices of finite rank greater than 1.** Let

$$A', A''; B', B''; \cdots ; P', P'' \tag{11}$$

be transformations defined respectively by matrices

$$\|a_{mk}'\|, \|a_{nl}''\|; \quad \|b_{mk}'\|, \|b_{nl}''\|; \cdots ;$$

$$\|p_{mk}'\|, \|p_{nl}''\|,$$

and let

$$A = A' \odot A'', \ B = B' \odot B'', \cdots , P = P' \odot P''.$$

Moreover, let

$$\{a_{mn}\}, \{b_{mn}\}, \cdots , \{p_{mn}\} \tag{12}$$

be any set of double sequences. If the number of the transformations (11) is $2r$, the matrix whose general element is

$$t_{mnkl} = a_{mn}a_{mk}'a_{nl}'' + b_{mn}b_{mk}'b_{nl}'' + \cdots + p_{mn}p_{mk}'p_{nl}'' \tag{13}$$

will be said to be of *rank r*. Theorem 2 may now be extended as follows.

THEOREM 5. *Let each of the transformations* (11) *be convergence-preserving and regular for null sequences, and in addition let each pair satisfy condition* (4). *Then, if the sequences* (12) *converge with respective limits a, b,* $\cdots$, *p, the transformation T defined by* (13) *is convergence-preserving for the class of double sequences which are simultaneously bounded A, bounded B,* $\cdots$, *and bounded P. A convergent double sequence of this class, with limit x, is carried into a sequence whose limit is* $(a+b+\cdots+p)x$; *hence the transformation T is regular for the class of sequences described if and only if we have* $a+b+\cdots+p=1$.

5. **Inclusiveness and equivalence.** For most of our subsequent work the following theorem is of primary importance.

THEOREM 6. *If we have**

$$A = A' \odot A'' \text{ and } B = B' \odot B'',$$

then we also have

$$B\cdot A = (B'\cdot A') \odot (B''\cdot A'').$$

The general element of the matrix defining the transformation $B\cdot A$ is

$$c_{mnrs} = \sum_{k=r,\,l=s}^{m,n} b'_{mk}b''_{nl}a'_{kr}a''_{ls} = \sum_{k=r}^{m} b'_{mk}a'_{kr}\sum_{l=s}^{n} b''_{nl}a''_{ls}.$$

Of these two sums the first is the general element c'_{mr} of the matrix defining the product $B'\cdot A'$, while the second is the general element c''_{ns} of the matrix defining $B''\cdot A''$.

From this we obtain at once

THEOREM 7. *If A' and A'' both have inverses, denoted respectively by $\mathfrak{A}'$ and $\mathfrak{A}''$, then $A = A' \odot A''$ has an inverse $\mathfrak{A} = \mathfrak{A}' \odot \mathfrak{A}''$.*

If the identical transformation for simple sequences be denoted by I', and that for double sequences by I, we have

$$I = I' \odot I'',$$

and hence

$$A\cdot\mathfrak{A} = (A'\cdot\mathfrak{A}') \odot (A''\cdot\mathfrak{A}'') = I' \odot I' = I,$$
$$\mathfrak{A}\cdot A = (\mathfrak{A}'\cdot A') \odot (\mathfrak{A}''\cdot A'') = I' \odot I' = I.$$

When every simple sequence evaluated by a transformation A' (i.e., for which the A'-transform converges) is evaluated, and assigned the same value, by a second transformation B', we say that B' *includes* A' and write

$$B' \supset A'.$$

The corresponding notion for double sequences we define in a similar way and indicate by a like relation.

THEOREM 8. *Let $A = A' \odot A''$ and $B = B' \odot B''$ be any two transformations of which the factor transformations, not necessarily regular, satisfy the conditions $B' \supset A'$ and $B'' \supset A''$ and A' and A'' have inverses. Then we have $B \supset A$ for the class of double sequences which are bounded B.*

* If A', B', A'', and B'' are defined by matrices which are not row-finite, we must add to the hypotheses of this theorem; it is sufficient to assume, in addition, absolute convergence of the series of elements in each row of the matrices defining respectively the four transformations. We may remark, however, that this restriction does little to impair the usefulness of the theorem.

Let the respective inverses be $\mathfrak{A}'$ and $\mathfrak{A}''$ and let $\{x_{mn}\}$ be any sequence for which $B\{x_{mn}\}$ is bounded and $A\{x_{mn}\}$ converges, let us say to x. By Theorems 6 and 7 we then have

$$\begin{aligned} B\{x_{mn}\} &= B\{\mathfrak{A}\cdot A\{x_{mn}\}\} \\ &= (B\cdot\mathfrak{A})\{A\{x_{mn}\}\} \\ &= [(B'\cdot\mathfrak{A}')\odot(B''\cdot\mathfrak{A}'')]\{A\{x_{mn}\}\}. \end{aligned} \tag{14}$$

From the relations $B'\supset A'$, $B''\supset A''$, it follows that the transformations $B'\cdot\mathfrak{A}'$ and $B''\cdot\mathfrak{A}''$ are both regular; hence $B\{x_{mn}\}$ converges to x and the theorem is proved.

As immediate consequences of this theorem we have several corollaries, of which the following concerning Cesàro summability is typical.

COROLLARY.* *If a double series is summable* (C, r, s) $(r, s > -1)$ *and bounded* (C, r', s') $(r' \geqq r, s' \geqq s)$, *the series is summable* (C, r', s') *to the same sum.*

When we have simultaneously $B'\supset A'$ and $A'\supset B'$ we say that A' and B' are *equivalent* and express this fact by the relation

$$B' \sim A'.$$

A similar terminology and notation will be used for the analogous relation between double sequence transformations.

From Theorem 8 we now obtain

THEOREM 9. *Let* $A = A'\odot A''$ *and* $B = B'\odot B''$ *be any two transformations of which the factor transformations, not necessarily regular, satisfy the conditions* $B'\sim A'$ *and* $B''\sim A''$ *and all have inverses. Then we have* $B\sim A$ *for the class of double sequences which are bounded* B (*or bounded* A).

That it is immaterial whether we say bounded B or bounded A is apparent from the relation (14) and its analogue expressing $A\{x_{mn}\}$ in terms of $B\{x_{mn}\}$, both of which are valid under the present hypotheses.

The following corollary is of interest:

COROLLARY. *The Cesàro* (C, r, s) *and Hölder* (H, r, s) $(r, s > -1)$ *definitions of summability are equivalent for the class of double series which are bounded* (C, r, s) (*or bounded* (H, r, s)).

We shall now state several theorems concerning inclusiveness and equivalence which do not depend upon the existence of inverses of the transforma-

* This corollary obviously includes Theorems 3, 4, and 5 of I as well as Theorem 1′ of Merriman, *Concerning the summability of double series of a certain type*, Annals of Mathematics, vol. 28 (1927), pp. 515–533.

tions involved. First we have two analogues of well known theorems on simple sequences.*

THEOREM 10. *Let A and B be any two transformations, not necessarily regular for any particular class of double sequences.*† *If there exists a transformation C, regular for all double sequences and satisfying the condition $B=C\cdot A$, we have $B\supset A$ for all double sequences.*

THEOREM 11. *Let A and B be any two transformations, not necessarily regular for any particular class of double sequences.*† *If there exist two transformations C and D, each regular for all double sequences, and together satisfying the conditions $B=C\cdot A$, $A=D\cdot B$, we have $B\sim A$ for all double sequences.*

Neither of these theorems is of any considerable interest, since the class of transformations regular for all double sequences is so restricted. The two following theorems, while more general than Theorems 8 and 9, still possess some degree of usefulness.

THEOREM 12. *Let $A=A'\odot A''$ and $B=B'\odot B''$ be any two transformations whose factors are not necessarily regular. If there exist regular transformations C' and C'' satisfying the conditions $B'=C'\cdot A'$, $B''=C''\cdot A''$, we have $B\supset A$ for the class of double sequences which are bounded B.*

THEOREM 13. *Let $A=A'\odot A''$ and $B=B'\odot B''$ be any two transformations whose factors are not necessarily regular. If there exist regular transformations C', C'', D', and D'' satisfying the conditions $B'=C'\cdot A'$, $B''=C''\cdot A''$, $A'=D'\cdot B'$, and $A''=D''\cdot B''$, we have $B\sim A$ for the class of double sequences which are bounded B (or bounded A).*

6. **Omission or adjunction of a row or column.** It is sufficient to consider the omission or adjunction of a row, since the situation with respect to columns is symmetrical. Let us set

$$y_{mn}=\sum_{k=1,l=1}^{m,n}a_{mnkl}x_{kl},\quad \bar{y}_{mn}=\sum_{k=1,l=1}^{m,n}a_{mnkl}x_{k+1,l},\tag{15}$$

and

$$y'_{mn}=\sum_{k=1,l=1}^{m,n}a_{m+1,n,k+1,l}x_{k+1,l}.\tag{16}$$

We seek to determine sufficient conditions that, when either $\{y_{mn}\}$ or $\{\bar{y}_{mn}\}$

* See Hurwitz, *Report on topics in the theory of divergent series*, Bulletin of the American Mathematical Society, vol. 28 (1922), p. 26.

† Other than the "class" of sequences all of whose elements are zero.

converges, the other will converge to the same limit. *If the transformation and sequence involved satisfy the condition*

$$\lim_{m,n\to\infty} \sum_{l=1}^{n} a_{mn1l}x_{1l} = 0, \tag{17}$$

we have

$$\lim_{m,n\to\infty} y_{mn} = \lim_{m,n\to\infty} y_{m+1,n} = \lim_{m,n\to\infty} y'_{mn}$$

whenever any one of these limits exists, and may therefore transfer our considerations from the pair y_{mn}, $\bar{y}_{mn}$ *to the pair* y'_{mn}, $\bar{y}_{mn}$.

Let A denote the first of transformations (15) *and* A_1 *the transformation* (16). If A and A_1 are equivalent for some class of sequences and if a sequence

$$\{x_{mn}\} \qquad (m,\, n = 1, 2, 3, \cdots) \tag{18}$$

of that class satisfies the condition (17), the sequences (18) and

$$\{x_{mn}\} \qquad (m = 2, 3, 4, \cdots;\, n = 1, 2, 3, \cdots) \tag{19}$$

are assigned the same value whenever either is evaluated. Thus by Theorem 11 we have

THEOREM 14. *Let A be any transformation, not necessarily regular for any particular class of double sequences, and let* (18) *be any sequence satisfying condition* (17). *If there exist two transformations C and D, each regular for all double sequences, and together satisfying the relations* $A_1 = C\cdot A$, $A = D\cdot A_1$, *then whenever either sequence* (18) *or* (19) *is evaluated by A, the other is assigned the same value.*

It may be remarked that when A is regular for the class of bounded sequences, any sequence (18) of the class described in Theorem 2 of I fulfills condition (17). For transformations of the product type we have by Theorem 13 the following:

THEOREM 15. *Let* $A = A' \odot A''$ *be any transformation, whose factors are not necessarily regular, and let* $A_1 = A_1' \odot A''$. *If there exist regular transformations* C' *and* D' *satisfying the conditions* $A_1' = C'\cdot A'$, $A' = D'\cdot A_1'$, *let* (18) *be any sequence satisfying the condition* (17) *and such that* (19) *is bounded A* (*or bounded* A_1). *Then if either sequence* (18) *or* (19) *is evaluated by A, the other is assigned the same value.*

The following corollary, obtained by taking A as the Cesàro transformation (C, r, s), may be of interest. Denoting by $(C, r)_1$ the transformation whose matrix is obtained from the (C, r) matrix by suppressing the first row and column, we may write

$$A = (C, r, s) = (C, r) \odot (C, s), \quad A_1 = (C, r)_1 \odot (C, s)$$

and obtain, for C' and D' of Theorem 15,

$$C' = (C, r)_1 \cdot (C, r)^{-1}, \quad D' = (C, r) \cdot [(C, r)_1]^{-1},$$

each of which is regular.* Hence we have the

COROLLARY. *Omission or adjunction of a row is permissible in the case of the Cesàro transformation* (C, r, s) *if the first row of* (18) *is bounded* (C, s) *and* (19) *is bounded* (C, r, s).

7. **Permutability; mutual consistency.** Two double sequence transformations A and B will be said to be permutable if and only if we have

$$B \cdot A = A \cdot B.$$

For the present all simple sequence transformations are understood to be defined by triangular matrices, and all double sequence transformations by four-dimensional matrices of analogous type.

THEOREM 16. *If we have*

$$A = A' \odot A'' \text{ and } B = B' \odot B''$$

and if A' *and* B', *and also* A'' *and* B'', *are permutable,* A *and* B *are permutable.*

By Theorem 6 we have

$$B \cdot A = (B' \cdot A') \odot (B'' \cdot A'') = (A' \cdot B') \odot (A'' \cdot B'') = A \cdot B.$$

It is evident that the arithmetic mean transformation for double sequences, M, defined by the matrix

$$a_{mnkl} = 1/(mn),$$

satisfies the equation

$$M = M' \odot M',$$

where M' indicates the arithmetic mean transformation for simple sequences. Thus we have

COROLLARY 1. *If* $A = A' \odot A''$ *is any transformation of which each factor is permutable with* M', A *is permutable with* M.

* See Hurwitz, loc. cit., p. 32, and Carmichael, *General aspects of the theory of summable series*, Bulletin of the American Mathematical Society, vol. 25 (1918), p. 118.

Let $T = A_1 + A_2 + \cdots + A_n$; then we clearly have $A \cdot T = A \cdot A_1 + A \cdot A_2 + \cdots + A \cdot A_n$ and obtain immediately

COROLLARY 2. *If $A = A' \odot A''$ and $A_i = A_i' \odot A_i''$ $(i = 1, 2, \cdots, n)$ are any $n+1$ transformations of which each of the factor transformations A_i' is permutable with A' and each of the A_i'' is permutable with A'', A and T are permutable.*

A third corollary, similar to this, can be stated for

$$A_1 + A_2 + \cdots + A_n + \cdots$$

whenever this symbol has a meaning.

Two double sequence transformations, A and B, will be said to be mutually consistent if, whenever each evaluates a double sequence, the values assigned to it are the same. We now turn to the problem of determining sufficient conditions for the mutual consistency of two transformations.

It is natural to call any transformation of the form

$$y_{mn} = f_{mn} x_{mn}$$

a multiplication. In addition to such transformations we are concerned with the Euler transformation for double sequences,

$$y_{mn} = \sum_{k=1,l=1}^{m,n} (-1)^{k+l} \frac{(m-1)!}{(m-k)!(k-1)!} \cdot \frac{(n-1)!}{(n-l)!(l-1)!} x_{kl},$$

which we denote by Δ. Evidently we have

$$\Delta = \Delta' \odot \Delta',$$

where Δ' stands for the Euler transformation for simple sequences, and hence

$$\Delta^2 = \Delta \cdot \Delta = (\Delta' \cdot \Delta') \odot (\Delta' \cdot \Delta') = I' \odot I' = I.$$

We may now prove at once six lemmas corresponding precisely to Hurwitz and Silverman's Lemmas 1–6*; these culminate in the last which is as follows.

LEMMA 3. *Any two double sequence transformations of which each is permutable with M, are permutable with each other.*

From this Lemma we obtain at once two theorems.

THEOREM 17. *Any two double sequence transformations of which each is permutable with M and regular for the class of bounded sequences, are mutually consistent for this class of sequences.*

* Hurwitz and Silverman, *On the consistency and equivalence of certain definitions of summability*, these Transactions, vol. 18 (1917), pp. 1–20.

The generality of this theorem can be extended somewhat by aid of Theorem 2 of I; this comment applies also to Theorem 19 and to Theorem 23 and its corollary.

THEOREM 18. *Let $A = A' \odot A''$ and $B = B' \odot B''$ be any two transformations of which each is permutable with M and all the factor transformations are regular; moreover, let $\{x_{mn}\}$ be any sequence which is evaluated by both A and B and is bounded $B \cdot A$ (or bounded $A \cdot B$). Then the values assigned to this sequence by A and B are the same.*

Further consequences of the above-mentioned lemmas are the following three theorems.

THEOREM 19. *If A and B are any two double sequence transformations of which each is permutable with M and regular for the class of bounded sequences, and if A evaluates a bounded sequence $\{x_{mn}\}$ to ξ and B evaluates a bounded sequence $\{y_{mn}\}$ to η, then $A \cdot B$ evaluates $\{x_{mn}+y_{mn}\}$ to $\xi+\eta$.*

THEOREM 20. *Let $A = A' \odot A''$ and $B = B' \odot B''$ be any two double sequence transformations of which each is permutable with M and all the factor transformations are regular; moreover, let $\{x_{mn}\}$ be any sequence which is evaluated by A, say to ξ, and is bounded $B \cdot A$ (or bounded $A \cdot B$), and let $\{y_{mn}\}$ be any second sequence, evaluated by B to η and bounded $B \cdot A$. Then $A \cdot B$ evaluates $\{x_{mn}+y_{mn}\}$ to $\xi+\eta$.*

THEOREM 21. *A necessary and sufficient condition that A be permutable with M is that there exist numbers f_{hi} $(h, i = 1, 2, 3, \cdots)$ such that we have*

$$
a_{mnkl} = \sum_{h=k,\,i=l}^{m,n} (-1)^{h+i-k-l} \frac{(m-1)!}{(m-h)!(h-k)!(k-1)!} \cdot \frac{(n-1)!}{(n-i)!(i-l)!(l-1)!} f_{hi}. \tag{20}
$$

More general sufficient conditions for mutual consistency are given by the three following theorems, which are free from any restriction on the shape of the matrices involved.*

THEOREM 22. *Any two double sequence transformations A and B are mutually consistent for a class of double sequences if there exists a third transformation C such that we have $C \supset A$ and $C \supset B$ for this class of sequences.*

THEOREM 23. *Any two transformations A and B are mutually consistent for the class of bounded sequences if there exist transformations C and D, each regular for this class of sequences and together satisfying the condition $C \cdot A = D \cdot B$.*

* See Hurwitz, loc. cit., p. 28.

Corollary. *Any two transformations A and B which are permutable and regular for the class of bounded sequences, are mutually consistent for this class of sequences.*

Theorem 24. *Let $A=A'\odot A''$ and $B=B'\odot B''$ be any two permutable transformations of which all the factor transformations are regular; moreover, let $\{x_{mn}\}$ be any sequence which is evaluated by both A and B and is bounded $B\cdot A$ (or bounded $A\cdot B$). Then the values assigned to this sequence by A and B are the same.*

8. **Transformations defined by certain matrices of infinite rank.** The following analogues of Hurwitz and Silverman's Theorem 1 and its corollary* are readily proved by a natural modification of their method.

Theorem 25. *If $f(z)=\alpha_0+\alpha_1 z+\alpha_2 z^2+\cdots$ is analytic within and on the boundary of the circle of unit radius about the origin and if we have $f(1)=1$, then the symbol $\alpha_0 I+\alpha_1 M+\alpha_2 M^2+\cdots$ defines a transformation A regular for the class of bounded sequences.*

Corollary. *The general element a_{mnkl} of the matrix corresponding to A is expressed in terms of $f(z)$ by a formula like (20) except for the replacement of f_{hi} by $f(1/(hi))$.*

From Theorem 2 of I it follows that the transformation A *may* be regular for some unbounded sequences. It is clear, however, that if $\alpha_0\neq 0$, the class defined by conditions (a) and (b) in Theorem 2 of I need not include any unbounded sequences. If α_0 is zero, every sequence $\{x_{mn}\}$ which is bounded M belongs to this class, as we shall now prove.

First of all, since the main diagonal elements of M' are all different from zero, we infer that each row and column of $\{x_{mn}\}$ is bounded M'. Let the general elements of M^r and $(M')^r$ be denoted respectively by

$$\overset{(r)}{a}_{mnkl} \text{ and } \overset{(r)}{a}_{mk};$$

then we have

$$a_{mnkl} = \lim_{P\to\infty} \sum_{r=1}^{P} \alpha_r \overset{(r)}{a}_{mnkl}.$$

Thus the transform of the lth column of $\{x_{mn}\}$ is, in the notation of Theorem 2 of I,

* See Hurwitz and Silverman, loc. cit.

$$\overset{nl}{u}_m = \sum_{k=1}^{m}\left(\lim_{P\to\infty} \sum_{r=1}^{P} \alpha_r \overset{(r)}{a}_{mnkl}\right) x_{kl}$$

$$= \lim_{P\to\infty} \sum_{r=1}^{P} \alpha_r \sum_{k=1}^{m} \overset{(r)}{a}_{mnkl} x_{kl}$$

(21) $$= \lim_{P\to\infty} \sum_{r=1}^{P} \alpha_r \overset{(r)}{a}_{nl} \sum_{k=1}^{m} \overset{(r)}{a}_{mk}\, x_{kl}.$$

But we have

$$\left| \sum_{k=1}^{m} \overset{(r)}{a}_{mk} x_{kl} \right| < B_l \qquad (m = 1, 2, 3, \cdots),$$

where B_l is a suitable constant. Hence the sum in (21) is at most equal to

$$B_l \sum_{r=1}^{P} |\alpha_r| \cdot |\overset{(r)}{a}_{nl}|,$$

and we have

(22) $$|\overset{nl}{u}_m| < B_l \lim_{P\to\infty} \sum_{r=1}^{P} |\alpha_r| \cdot |\overset{(r)}{a}_{nl}| .$$

It still remains to show that this bound converges to zero with $1/n$. Since $\sum_{r=1}^{\infty} |\alpha_r|$ converges and $|\overset{(r)}{a}_{nl}| \leqq 1$, the limit in (22) exists uniformly with respect to n; and, M' being regular,

$$\lim_{n\to\infty} \sum_{r=1}^{P} |\alpha_r| \cdot |\overset{(r)}{a}_{nl}|$$

exists and equals zero for each P. Hence we have for this sum

$$\lim_{n\to\infty} \lim_{P\to\infty} = \lim_{P\to\infty} \lim_{n\to\infty} = 0,$$

which was to be proved. The corresponding condition on columns of $\{x_{mn}\}$ may similarly be shown to hold. We formulate the result now established in

THEOREM 26. *If $f(z) = \alpha_1 z + \alpha_2 z^2 + \cdots$ is analytic within and on the boundary of the circle of unit radius about the origin and if we have $f(1) = 1$, then the symbol $\alpha_1 M + \alpha_2 M^2 + \cdots$ defines a transformation A regular for the class of sequences which are bounded M.*

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